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Completeness for a New Logic of Conditionals

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The language $\mathcal{L}$ is the language of propositional logic extended with the two-place sentential operator $\Box\!\!\rightarrow$. We can think of this as expressing the subjunctive conditional, and so read $\phi \Box\!\!\rightarrow \psi$ as saying that had it been that ϕ, it would have been that ψ. Most of what we will say in the following, though, will apply equally well to the indicative reading.

The system **B1** is a kind of minimal conditional logic from Burgess (1981). That system has two rules of inference:

MP $\quad \phi, \phi \supset \psi \vdash \psi$
SLE $\quad \vdash (\phi \Box\!\!\rightarrow \psi) \supset (\phi^* \Box\!\!\rightarrow \psi^*)$ when $\vdash \phi \equiv \phi^*$ and $\vdash \psi \equiv \psi^*$

The basic axioms are all instances of the following:

PL $\quad \phi$ when ϕ is a theorem of classical propositional logic
A1 $\quad \phi \Box\!\!\rightarrow \phi$
A2 $\quad (\phi \Box\!\!\rightarrow \psi \wedge \gamma) \supset (\phi \Box\!\!\rightarrow \psi)$
A3 $\quad (\phi \Box\!\!\rightarrow \psi \wedge \gamma) \supset (\phi \wedge \psi \Box\!\!\rightarrow \gamma)$
A4 $\quad (\phi \Box\!\!\rightarrow \psi) \wedge (\phi \Box\!\!\rightarrow \gamma) \supset (\phi \Box\!\!\rightarrow \psi \wedge \gamma)$
A5 $\quad (\phi \Box\!\!\rightarrow \gamma) \wedge (\psi \Box\!\!\rightarrow \gamma) \supset (\phi \vee \psi \Box\!\!\rightarrow \gamma)$

Other systems can then be formed by extending **B1** with various axioms. Our main interest will be in the following:

D $\quad (\phi \vee \psi \Box\!\!\rightarrow \gamma) \supset (\phi \Box\!\!\rightarrow \gamma) \vee (\psi \Box\!\!\rightarrow \gamma)$
RM $\quad (\phi \Box\!\!\rightarrow \gamma) \wedge \neg(\phi \Box\!\!\rightarrow \neg\psi) \supset (\phi \wedge \psi \Box\!\!\rightarrow \gamma)$
CEM $\quad (\phi \Box\!\!\rightarrow \psi) \vee (\phi \Box\!\!\rightarrow \neg\psi)$

Fixing **B1** as the background logic, these axioms are in reverse order with respect to strength. D is a principle that has not been much discussed, and so has no official designation. We are going to call it disjunction distribution. RM is a principle called rational monotonicity. CEM is the counterfactual law of the excluded middle. The systems that result by adding these axioms to **B1** will be called **B2**, **B3**, and **B4** respectively.

Three of these four systems have prominent defenders in the literature. Robert Stalnaker (1968) was an early defender of a system that he calls **C2**, which is the result of adding weak centering to **B4**:

WC $(\phi \mathbin{\Box\!\!\rightarrow} \psi) \supset (\phi \supset \psi)$

The weakest system that David Lewis (1971, 1973) considers is a system that he calls **V**, which is the same as our **B3**. That system is notable because it is the weakest system that can be modeled using his systems of spheres. Lewis ultimately endorses a slightly stronger system **VC** that adds both weak centering and strong centering to **B3**. Strong centering is the principle that:

SC $(\phi \wedge \psi) \supset (\phi \mathbin{\Box\!\!\rightarrow} \psi)$

Finally, John Pollock (1975, 1976a, 1976b) rejects RM, and so accepts a logic that he calls **SS**. That system is formed by adding both SC and WC to **B1**. Setting aside strong and weak centering, then, we can think of Stalnaker as accepting **B4**, Lewis as accepting **B3**, and Pollock as accepting **B1**.

My own view is that **B2** is in fact the right logic for counterfactuals. My reasons for thinking this are, in broad outline, as follows: First, I side with Lewis against Stalnaker on CEM. It is neither true that had I flipped a coin one minute ago, it would have landed heads, nor is it true that had I flipped a coin one minute ago, it would have landed tails. This rules out **B4**.

Second, I side with Pollock against Lewis on rational monotonicity, though I also reject Pollock's alleged counterexamples. I deny the validity of RM because this strikes me as the most natural way to resolve the paradox of counterfactual tolerance.[1] Since I reject RM, this rules out **B3**.

Finally, while I side with Pollock against Lewis, his preferred **B1** invalidates D. But D is obviously valid! This is a fact that Pollock seems to have either failed to notice, or failed to fully appreciate. For example, suppose that:

Had it either rained or snowed, Sophie would have been pleased. (1)

1. The paradox is introduced in my (2020) and will be briefly sketched in §1.

From this it would seem to follow that *at least one* of the following is true:

Had it rained, Sophie would have been pleased. (2)

Had it snowed, Sophie would have been pleased. (3)

These two claims cannot both be false. But if Pollock is right and **B1** is the right logic for counterfactuals, this is simply not so: The truth of (1) is entirely compatible with the joint falsehood of (2) and (3). This strikes me as absurd, which rules out **B1**.

Here is the plan for the rest of this paper. We will start by briefly sketching the paradox of counterfactual tolerance in §1. This will help motivate our interest in systems that invalidate RM.

We will then introduce accessibility models for **B2** in §2. One of the difficulties is that counterfactual accessibility is generally thought to be reflexive and transitive, following Lewis (1971) and Burgess (1981). There are consistent sentence of **B2**, though, whose only models are non-transitive. This means that existing approaches to proving completeness cannot be easily extended to **B2**. Furthermore, it is not enough to simply deny transitivity, because doing so invalidates not just **B2**, but also the weaker **B1**. We thus need to replace transitivity with a weaker condition, one that I call weak chirality. **B2** is then sound and complete with respect to the class of models that are reflexive, connected, and weakly chiral.

The main results in this paper will be proved in the last two sections. §3 proves soundness, which is somewhat less straightforward than usual. §4 proves completeness. Since the models we will construct are finite, we will thereby also prove decidability.

The primary focus of this paper is technical rather than philosophical. I have elsewhere argued at length for the validity of D and against the validity of RM. Here, we will only say enough about those issues to motivate a technical interest in **B2**. Readers interested in further philosophical discussion are invited to see my (2020). This present paper, in fact, might be thought of as a kind of technical companion to that paper.

1. The Paradox of Counterfactual Tolerance

Plank lengths are incredibly small. You would quite literally need a billion trillion of them just to span that diameter of a proton. Now suppose that Barak Obama is in fact h plank lengths tall. It would then seem that the following claims are true:

Tolerance: For all positive integers $n > h$, it is false that had Obama been at least n plank lengths, he would not have been at least $n + 1$ plank lengths.

Boundedness: There are positive integers $j, k > h$ such that had Obama been at least j plank lengths, he would not have been at least k plank lengths.

Heights: For all positive integers n, had Obama been at least $n + 1$ plank lengths, he would have been at least n plank lengths.

Tolerance says that it is *false* that had Obama been at least seven feet, he would not have been at least one Plank length taller than seven feet, and likewise for other heights.[2] Boundedness will be true if, for example, had Obama been at least seven feet, he would not have been at least a thousand feet. Heights says that had Obama been at least seven feet and one plank length, he would thereby have been at least seven feet, and likewise for other heights.

Given these three attractive claims, we can prove a flat contradiction using any system extending **B3**. First, we observe that the following axiom is valid in not only **B3**, but any system extending **B1**.[3]

$$\text{E4} \quad (\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \wedge \psi \mathrel{\Box\!\!\rightarrow} \gamma) \supset (\phi \mathrel{\Box\!\!\rightarrow} \gamma)$$

This principle is often called limited transitivity. Besides being derivable, the principle is also compelling in its own right, and so often taken as basic in various systems.[4]

This gives us everything we need to sketch the paradox. Let n express the claim that Obama is at least n inches tall, $n + 1$ express the claim that Obama is at least

2. If you accept the duality of would and might counterfactuals, you could also read tolerance as saying that had Obama been at least seven feet, he *might* have been at least one Plank length taller than seven feet, and likewise for other heights.
3. This will be proved later on in lemma 4.1.
4. See Fine (2012) for example.

$n + 1$ inches tall, and so on. We then reason as follows:

1.	$n \mathrel{\Box\!\!\rightarrow} \neg k$	boundedness
2.	$\neg(n \mathrel{\Box\!\!\rightarrow} \neg(n+1))$	tolerance
3.	$n+1 \mathrel{\Box\!\!\rightarrow} n$	heights
4.	$n \wedge (n+1) \mathrel{\Box\!\!\rightarrow} \neg k$	1,2, RM
5.	$(n+1) \wedge n \mathrel{\Box\!\!\rightarrow} \neg k$	4, SLE
6.	$n+1 \mathrel{\Box\!\!\rightarrow} \neg k$	3, 5, E4

This argument is paradoxical because it can be iterated. In particular, after $k - n - 1$ applications, we get:

$$k - 1 \mathrel{\Box\!\!\rightarrow} \neg k \tag{4}$$

But tolerance tells us that

$$\neg(k - 1 \mathrel{\Box\!\!\rightarrow} \neg k) \tag{5}$$

and so we have a flat contradiction. Since we have reasoned to paradox, something has to go: Either one of the attractive claims we started with is false, or **B3** is invalid.

There are many strategies for responding to the paradox, several of which I consider at length in my (2020).[5] For present purposes, we will simply note that denying RM is one natural response, and is in fact my own preferred solution. We thus have good reason to be interested in systems that invalidate RM.

2. Accessibility Models

Counterfactuals and other conditionals are often modeled using a three-place accessibility relation $Rxab$ on worlds, which we will read as saying that a is **accessible** from b relative to x.[6] A world a is **counterfactually possible** relative to x if there is some b such that $Rxab$, a condition that we will express with the notation Pxa. Finally, when $Rxab$ and not $Rxba$, we will say that a is **strictly accessible** from b relative to x,

5. Defenders of CEM, for example, will be inclined to deny tolerance. This response has its appeal but, as I point out in the referenced paper, denying tolerance on the basis of the CEM only kicks the can down the road. This is because there is a structurally similar paradox that can be staged using only might counterfactuals, and affirming the CEM is of no help in that case.

6. Accessibility models for counterfactuals were introduced by Lewis (1971), before being generalized in important ways by Burgess (1981).

and will express that condition with $Sxab$. When the x is either arbitrary or clear from context, we will generally drop explicit reference to it, and so say that a is accessible from b, that a is counterfactually possible, and that a is strictly accessible from b.

Definition 2.1. *An* ***accessibility model*** *for $\mathcal{L}$ is a triple $\mathcal{M} = \langle W, R, V\rangle$ consisting of a non-empty set of worlds, a three-place accessibility relation on worlds, and a valuation function assigning atomic sentences to sets of worlds. Truth at a world is defined recursively:*

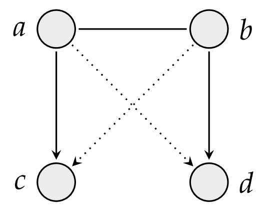
$x \models p$	*iff*	$x \in V(p)$
$x \models \neg\phi$	*iff*	$x \not\models \phi$
$x \models \phi \vee \psi$	*iff*	*either* $x \models \phi$ *or* $\models \psi$
$x \models \phi \wedge \psi$	*iff*	$x \models \phi$ *and* $\models \psi$
$x \models \phi \supset \psi$	*iff*	*either* $x \not\models \phi$ *or* $\models \psi$
$x \models \phi \mathrel{\Box\!\!\rightarrow} \psi$	*iff*	*for all* $a \models \phi$ *such that* Pxa*, there* a $b \models \phi$ *such that* $Rxba$ *and, for all* c *such that* $Rxcb$*, if* $c \models \phi$ *then* $c \models \psi$

A sentence is true in a model when true at every world.

Definition 2.2. *In the table below, the lefthand column lists various frame conditions, which are defined using the corresponding expression on the right.*

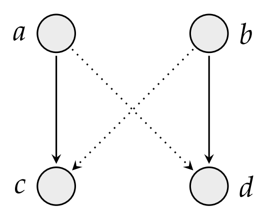
reflexive	$\forall x \forall a(Pxa \supset Rxaa)$
connected	$\forall x \forall a \forall b(Pxa \wedge Pxb \supset Rxba \vee Rxab)$
transitive	$\forall x \forall a \forall b \forall c(Pxa \wedge Rxcb \wedge Rxba \supset Rxca)$
weakly chiral	$\forall x \forall a \forall b \forall c \forall d(Pxa \wedge Pxb \wedge Rxca \wedge Rxdb \wedge (Rxab \vee Rxba) \supset Rxad \vee Rxbc)$
strongly chiral	$\forall x \forall a \forall b \forall c \forall d(Pxa \wedge Pxb \wedge Rxca \wedge Rxdb \supset Rxad \vee Rxbc)$
weakly centered	$\forall x(Rxxx)$
strongly centered	$\forall x \forall a(Rxax \supset a = x)$
standard	*reflexive and weakly chiral*
total	*reflexive, connected, and transitive*

Weak and strong chirality are new, and so best described with a pair of diagrams. Weak chirality says: Take any a and b that are counterfactually possible. Suppose furthermore that c is accessible from a and that d is accessible from b. These accessibility relation are illustrated below with solid arrows:



Now suppose that a and b are connected, in the sense that either a is accessible from b or b is accessible from a. In the diagram, the fact that a and b are connected is represented by the solid bar between a and b. Given these assumptions, weak chirality tells us that either c is accessible from b or d is accessible from a. Those relations are represented with the crossing dotted arrows.

Strong chirality is just like weak chirality, except that there is no antecedent requirement that a and b are connected. That is: Let a and b be counterfactually possible, and suppose that c is accessible from a and d is accessible from b. These relations are again represented with solid arrows:



Strong chirality then tells us that either c is accessible from b or d is accessible from a. We are referring to these two frame conditions as weak and strong chirality because, when drawing these sorts of diagrams, the dotted arrows form a characteristic X.

What we are calling standard relations are relations that are reflexive and weakly chiral. These have several useful properties, some of which are listed below. The proofs are all straightforward, and so left to the reader.

Definition 2.3. *Say that b **covers** a relative to x when for all c, if $Rxca$, then $Rxcb$. Using $Cxab$ to express the fact that b covers a relative to x, we then have:*

weakly covering $\forall x \forall a \forall b (Rxab \vee Rxba \supset Cxab \vee Cxba)$
strongly covering $\forall x \forall a \forall b (Cxab \vee Cxba)$

Observation 2.1. *If a relation is standard, then it is weakly covering.*

Observation 2.2. *If a relation is standard and connected, then it is strongly covering.*

Observation 2.3. *A relation is standard and connected iff it is reflexive and strongly chiral.*

Observation 2.4. *Let R be a standard relation and S the corresponding strict relation. In that case, $Cxba$ whenever $Sxba$.*

Observation 2.5. *If a relation R is standard, then the corresponding strict relation S is transitive.*

We will now describe a number of results, all of which will be shown, either directly or indirectly, in what follows. The first is that **B1** is sound and complete with respect to the class of all standard models. This generalizes an existing result from Burgess (1981), who shows that the system is sound and complete with respect to all models that are reflexive and *transitive*. Every transitive relation is weakly chiral, so completeness is immediate. The main observation, then, is that **B1** is in fact *sound* with respect to the larger class of models. That this is the case is not completely obvious, so we will say more about this in §3.

The main contribution of this paper will be to show that **B2** is sound and complete with respect to the class of all *connected* standard models (or equivalently, given observation 2.3, the class of all reflexive and strongly chiral models). This requires a different proof procedure than the one used by Burgess, since there are consistent sentences of **B2** that have only non-transitive models.

Our procedure for constructing models is quite general, and can be use to show that various extensions of **B2** are complete with respect to more restricted classes. We could show, for example, that **B3** is sound and complete with respect to the class of all standard models that are connected and transitive, or that **B4** is sound and completely with respect to the class of all such models that are, in addition, well-founded. These results are already well-know from Lewis (1971). His proof of them, though, is highly indirect, taking an unnecessary detour through an alternative model theory. The procedure used here, in contrast, will not require any such detours.

Putting all of these observations together gives us the following table associating systems with the corresponding class of standard models relative to which they are sound and complete.

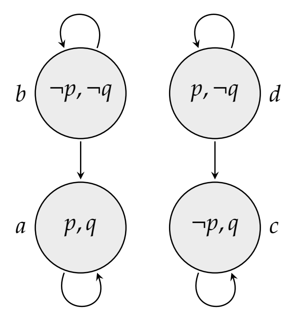
System	Condition
B1	all
B2	connected
B3	connected and transitive
B4	connected, transitive, and well-founded

This table illustrates the importance of dropping the usual assumption of transitivity.

If we require accessibility models to be transitive, then we face a kind of artificial choice between **B3** and **B1**. We thus have to either accept the validity of RM or deny the validity of D. Once we replace transitivity with weak chirality, though, we can resolve the dilemma, since we can model the intermediate system **B2**.

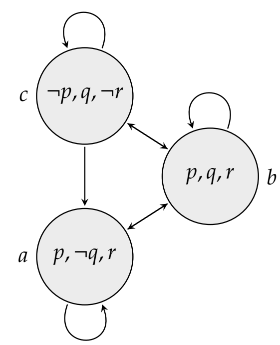
You might also think of the above table as establish a certain natural connection between modal logic and counterfactual logic. In particular, the validity of RM depends on transitivity, and so RM is a kind of counterfactual analogue of the modal principle S4. This is perhaps not surprising, given the structural between the paradox we sketched in §1 and the sorts of tolerance arguments often staged against S4 in various contexts.[7]

Using these facts about completeness, we now prove two of the major claims we made in the introduction. We can show that D is in fact invalid in **B1**, and that RM is in fact invalid in **B2**. Letting all accessibility relations be relative to the same fixed world x, the first fact can be shown with the following countermodel:



This is a standard model in which $x \not\models (p \vee \neg p \mathrel{\Box\!\!\rightarrow} q) \supset (p \mathrel{\Box\!\!\rightarrow} q \vee \neg p \mathrel{\Box\!\!\rightarrow} q)$, and so D is invalid in **B1**. Letting all accessibility relations again be relative x, the countermodel verifying the second fact is:

7. See Williamson (1992, 2000), Chandler (1976), Salmon (1979), and the large literature that followed.



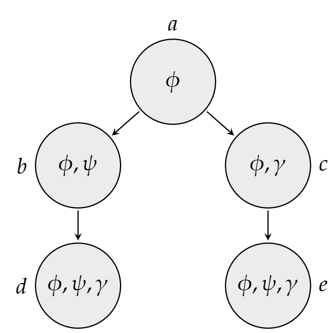
This is a connected standard model in which $x \not\models (p \mathrel{\Box\!\!\rightarrow} r \wedge \neg(p \mathrel{\Box\!\!\rightarrow} \neg q)) \supset (p \wedge q \mathrel{\Box\!\!\rightarrow} r)$, and so RM fails in **B2**.

3. Soundness

In this section, we are going to show how to prove the soundness results summarized in §2. Soundness proofs are generally routine, and this is mostly true in the present case. Weak and strong chirality, though, are unfamiliar conditions, and so it will be worth working through two of the less obvious induction clauses.

Theorem 3.1. ***B1*** *is sound with respect to the class of all standard models.*

Proof. Here, we are just going to show how to prove the soundness of A4. To that end, suppose $x \models (\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \mathrel{\Box\!\!\rightarrow} \gamma)$ and consider any $a \models \phi$ such that Pxa. By reflexivity, there is in fact such a world, with there also being further worlds arranged as follows, with all accessibility relations being relative to x. The reflexive arrows have been omitted to simplify the diagram.



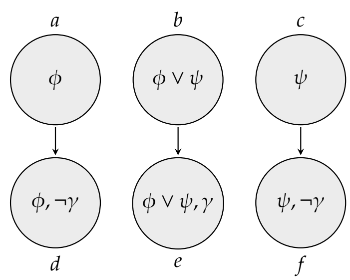
Furthermore, $f \models \psi$ for every world $f \models \phi$ such that either $Rxfb$ or $Rxfe$, and $g \models \gamma$ for every world $g \models \phi$ such that either $Rxgc$ or $Rxgd$.

Say that a world f is verifying if $Rxfa$ and, whenever $g \models \phi$ and $Rxgf$, $g \models \psi \wedge \gamma$. Since accessibility is standard, we know that it is also weakly covering by observation 2.1. To finish the proof then, all we need to do is show that there is a verifying world given any possible configuration of covering relations.

Suppose that d covers b. In that case, b is verifying, so suppose instead that b covers d. In that case, if a also covers b, then d is verifying. So we must suppose that b covers a. By parallel reasoning, we must also assume that e covers c and that c covers a. But in that case, a is verifying. So given any possible configuration of covering relations, there is a verifying world. ■

Theorem 3.2. ***B2*** *is sound with respect to the class of all connected standard models.*

Proof. Suppose for reductio that $x \not\models (\phi \vee \psi \mathrel{\Box\!\!\rightarrow} \gamma) \supset (\phi \mathrel{\Box\!\!\rightarrow} \gamma) \vee (\psi \mathrel{\Box\!\!\rightarrow} \gamma)$ for some x. We thus have



with the reflexive arrows again suppressed. Furthermore, a is such that for every $g \models \phi$ such that $Rxga$, there is an $h \models \phi \wedge \neg\gamma$ such that $Rxhg$. The world c is such that for every $g \models \psi$ such that $Rxgc$, there is an $h \models \psi \wedge \neg\gamma$ such that $Rxhg$. And finally, the world e is such that $h \models \gamma$ whenever $h \models \phi \vee \psi$ and $Rxhe$.

Our diagram is incomplete, since our accessibility relation is connected. So how can it be filled in? The first observation is that either not $Rxea$ or not $Rxec$ since otherwise, there will be some $h \models \phi \vee \psi$ such that $Rxhe$ and $h \models \neg\gamma$, which is contrary to assumption. This means that by connectedness, either $Sxae$ or $Sxce$. By observation 2.4, this means that either $Cxae$ or $Cxce$ and so either $Rxde$ or $Rxfe$. But this is also contrary to assumption, so we have our result. ■

4. Completeness

The most familiar canonical constructions, like those from Henkin (1949), build a single canonical model every consistent sentence of the target language. What we are going to do instead is assign each sentence of $\mathcal{L}$ a type and then, for each type, build a corresponding canonical model.

Definition 4.1. *Fix an enumeration $p_1, p_2, \ldots$ of the atomic sentence of $\mathcal{L}$. The **atomic type** of a sentence ϕ is the smallest n such that $p_1, \ldots, p_n$ includes all the atomic sentences in ϕ.*

Definition 4.2. *The **modal depth** of a sentence ϕ is given by $f(\phi)$, where this is defined recursively with:*

$$
\begin{aligned}
&f(\phi) = 0 \textit{ when } \phi \textit{ is an atom.}\\
&f(\neg\phi) = f(\phi) = n\\
&f(\phi \wedge \psi) = f(\phi \vee \psi) = f(\phi \supset \psi) = max(f(\phi), f(\psi))\\
&f(\phi \mathrel{\Box\!\!\rightarrow} \psi) = max(f(\phi), f(\psi)) + 1
\end{aligned}
$$

Definition 4.3. *The **type** of a sentence is $t = \langle n, m\rangle$, where n is the atomic type and m is the modal depth.*

Definition 4.4. *The **states** of type $\langle n, m\rangle$ are the members of $Y^{n,m}$, where this set is defined recursively with:*

$$
\begin{aligned}
X^{n,0} &= \textit{the set of atomic sentence of type } \langle n, 0\rangle\\
Y^{n,m} &= \textit{the set of consistent conjunctions } \phi_1 \wedge \cdots \wedge \phi_n \textit{ with } \phi_i \textit{ being}\\
&\quad\ \textit{either } \psi_i \textit{ or } \neg\psi_i \textit{ for the enumerated } \psi_i \in X^{n,m}\\
X^{n,m+1} &= \textit{the union of the } X^{n,m} \textit{ and all sentences of the form } \phi \vee \psi \mathrel{\Box\!\!\rightarrow}\\
&\quad\ \neg\psi \textit{ for } \phi, \psi \in Y^{n,m}
\end{aligned}
$$

Note that in the above construction, we always fix an enumeration of the relative atoms in $X^{n,m}$ in order to form the states in $Y^{n,m}$. This is important because it ensures that numerically distinct states are always logically inconsistent.

We now have almost everything needed to build our canonical models. As a final bit of preamble, we are going to fix a function mapping each state x to a maximal consistent set x^* such that $x \in x^*$. We then institute the following shorthand:

$$
\begin{aligned}
&a \trianglelefteq_x b \quad \text{iff} \quad x^* \vdash \neg(a \vee b \mathrel{\Box\!\!\rightarrow} \neg a)\\
&a \triangleleft_x b \quad \text{iff} \quad a \trianglelefteq_x b \text{ and } a \ntrianglelefteq_x b
\end{aligned}
$$

When the x is arbitrary or clear from context, we will drop the corresponding subscript, and so just write $a \trianglelefteq b$ and $a \triangleleft b$.

Definition 4.5. *For every type t, the corresponding canonical model $\mathcal{M}^t$ is constructed as follows:*

$$\begin{array}{lcl} W & = & \{x \mid x \text{ is a state of type } t\} \\ Rxab & \textit{iff} & a \trianglelefteq_x b \\ V(p) & = & \{x \in W \mid p \in x^*\} \end{array}$$

Proposition 4.1 (Deduction Theorem)**.** $\phi \vdash \psi$ *iff* $\vdash \phi \supset \psi$.

Proof. The proof is the same as in the propositional case, and so left to the reader. ∎

Lemma 4.1. *The following schemas are all valid in any system extending **B1**:*

E1 $(\phi \mathrel{\Box\!\!\rightarrow} \psi) \supset (\phi \mathrel{\Box\!\!\rightarrow} \gamma)$ *when* $\vdash \psi \supset \gamma$
E2 $\phi \mathrel{\Box\!\!\rightarrow} \psi$ *when* $\vdash \phi \supset \psi$
E3 $(\phi \mathrel{\Box\!\!\rightarrow} \gamma) \supset (\phi \vee \psi \mathrel{\Box\!\!\rightarrow} \gamma \vee \psi)$
E4 $(\phi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\phi \wedge \psi \mathrel{\Box\!\!\rightarrow} \gamma) \supset (\phi \mathrel{\Box\!\!\rightarrow} \gamma)$
E5 $(\phi \vee \psi \vee \gamma \mathrel{\Box\!\!\rightarrow} \neg\psi \wedge \neg\gamma) \supset (\phi \vee \psi \mathrel{\Box\!\!\rightarrow} \neg\psi)$
E6 $(\phi \vee \psi \mathrel{\Box\!\!\rightarrow} \neg\psi) \wedge (\phi \vee \gamma \mathrel{\Box\!\!\rightarrow} \neg\gamma) \supset (\phi \vee \psi \vee \gamma \mathrel{\Box\!\!\rightarrow} \neg\psi \wedge \neg\gamma)$
when $\psi \vdash \neg\gamma$
E7 $(\psi \mathrel{\Box\!\!\rightarrow} \gamma) \supset (\phi \vee \psi \mathrel{\Box\!\!\rightarrow} \gamma)$ *when* $\phi \vdash \gamma$
E8 $(\phi \mathrel{\Box\!\!\rightarrow} \neg\phi) \supset (\phi \mathrel{\Box\!\!\rightarrow} \psi)$

Proof. To show E1, let $\vdash \psi \supset \gamma$. Then:

1. $\phi \mathrel{\Box\!\!\rightarrow} \psi$
2. $\phi \mathrel{\Box\!\!\rightarrow} \gamma \wedge \psi$ 1, SLE
3. $\phi \mathrel{\Box\!\!\rightarrow} \gamma$ 2, A2

To show E2:

1. $\phi \mathrel{\Box\!\!\rightarrow} \phi$ A1
2. $\phi \mathrel{\Box\!\!\rightarrow} \psi$ 1, E1

To show E3:

1. $\phi \mathrel{\Box\!\!\rightarrow} \psi$
2. $\phi \mathrel{\Box\!\!\rightarrow} \psi \vee \gamma$ 1, E1
3. $\gamma \mathrel{\Box\!\!\rightarrow} \psi \vee \gamma$ E2
4. $\phi \vee \gamma \mathrel{\Box\!\!\rightarrow} \psi \vee \gamma$ 2, 3, E5

To show E4:

1. $\phi \boxright \psi$
2. $\phi \wedge \psi \boxright \gamma$
3. $(\phi \wedge \psi) \vee (\phi \wedge \neg\psi) \boxright \gamma \vee (\phi \wedge \neg\psi)$ 2, E3
4. $\phi \boxright \gamma \vee (\phi \wedge \neg\psi)$ 3, SLE
5. $\phi \boxright \psi \wedge (\gamma \vee (\phi \wedge \neg\psi))$ 1,4, A4
6. $\phi \boxright \gamma$ 5, SLE

To show E6, let $\psi \vdash \neg\gamma$. Then:

1. $\phi \vee \psi \boxright \neg\psi$
2. $\phi \vee \gamma \boxright \neg\gamma$
3. $\phi \vee \psi \vee \gamma \boxright \neg\psi \vee \gamma$ 1, E1
4. $\phi \vee \psi \vee \gamma \boxright \psi \vee \neg\gamma$ 2, E1
5. $\phi \vee \psi \vee \gamma \boxright (\neg\psi \vee \gamma) \wedge (\neg\gamma \vee \psi)$ 3, 4, A4
6. $\phi \vee \psi \vee \gamma \boxright \neg\psi \wedge \neg\gamma$ 5, SLE

To show E7, let $\phi \vdash \gamma$. Then:

1. $\psi \boxright \gamma$
2. $\phi \boxright \gamma$ E2
3. $\phi \vee \psi \boxright \gamma$ 2, 1, A5

To show E8:

1. $\phi \boxright \neg\phi$
2. $\phi \boxright \phi$ 2, A1
3. $\phi \boxright \phi \wedge \neg\phi$ 1,2 A4
4. $\phi \boxright \psi$ 4, E1

■

Proposition 4.2. $\trianglelefteq$ *is standard and connected.*

Proof. By observation 2.3, we can demonstrate the proposition by showing that $\trianglelefteq$ is reflexive and strongly chiral. To show reflexivity, let a and b be any state such that $a \neq b$, from which it follows that $b \vdash \neg a$. Then:

1. $a \vee a \boxright \neg a$
2. $a \boxright \neg a$ 1, SLE
3. $a \vee b \boxright \neg a \vee b$ 2, E3
4. $a \vee b \boxright \neg a$ 3, SLE

This gives us reflexivity by contraposition. To show strong chirality:

1.	$(a \vee d \mathrel{\Box\!\!\to} \neg a) \wedge (c \vee b \mathrel{\Box\!\!\to} \neg c)$	
2.	$a \vee d \vee c \vee b \mathrel{\Box\!\!\to} \neg a \wedge \neg c)$	1, E6
3.	$a \vee b \vee c \vee d \mathrel{\Box\!\!\to} \neg a \wedge \neg c)$	2, SLE
4.	$(a \vee b \mathrel{\Box\!\!\to} \neg a \wedge \neg c) \vee (c \vee d \mathrel{\Box\!\!\to} \neg a \wedge \neg c)$	3, D
5.	$(a \vee b \mathrel{\Box\!\!\to} \neg a) \vee (c \vee d \mathrel{\Box\!\!\to} \neg c)$	4, E1

Strong chirality then follows by contraposition and the maximality of x^*. ∎

Proposition 4.3. *Every sentence ϕ of type $\langle n, m\rangle$ is logically equivalent to a sentence ψ of type $\langle i, j\rangle$ whenever $i \geq n$ and $j \geq j$.*

Proof. Given any ϕ of type $\langle n, m\rangle$, we can find a logically equivalent ψ_1 of type $\langle n+1, m\rangle$ by using $\phi \wedge (\phi \vee p_{n+1})$, and a logically equivalent ψ_2 of type $\langle n, m+1\rangle$ by using $\phi \wedge (\phi \vee (\phi \mathrel{\Box\!\!\to} \phi))$. ∎

Lemma 4.2. *Let ϕ and ψ be sentences of type t. Then $\phi \mathrel{\Box\!\!\to} \psi$ is logically equivalent to $\bigvee_i \bigwedge_j \left(a_i \vee b_j \mathrel{\Box\!\!\to} \neg b_j\right)$, where the a_i and b_j are the states of type t such that $a_i \vdash \phi \wedge \psi$ and $b_j \vdash \phi \wedge \neg\psi$ respectively.*

Proof. The proof is by induction. For the base case, suppose ϕ and ψ are both of type $\langle m, 0\rangle$ and consider the sentence $\phi \mathrel{\Box\!\!\to} \psi$. This is equivalent to

$$(\phi \wedge \psi) \vee (\phi \wedge \neg\psi) \mathrel{\Box\!\!\to} \neg(\phi \wedge \neg\psi) \tag{6}$$

by substitution. Every sentence of type $\langle m, 0\rangle$ is equivalent to a sentence of type $\langle m, 0\rangle$ in disjunctive normal form in which every conjunction is maximal with respect to sentences of type $\langle m, 0\rangle$. As such, (6) is equivalent to

$$\bigvee_i (a_i) \vee \bigvee_j (b_j) \mathrel{\Box\!\!\to} \bigwedge_j \neg(b_j) \tag{7}$$

where the a_i and b_j are as described. The only thing left to show is that this is equivalent to the target sentence:

$$\bigvee_i \bigwedge_j \left(a_i \vee b_j \mathrel{\Box\!\!\to} \neg b_j\right) \tag{8}$$

The proof from (7) to (8) uses repeated applications of D and E5. The other direction uses disjunctive syllogism and repeated applications of E6 and E7. This gives us the base case. The induction step is essentially the same, with the exception that we use the induction hypothesis when showing that (6) is equivalent to (7). So the full result follows. ∎

Observation 4.1. *If ϕ is a state of type t and x is a state of type t, then $x \vdash \phi$ iff $x^* \vdash \phi$.*

Lemma 4.3. *Let ϕ and ψ be sentences of type t. Then the following are equivalent:*

$$x \vdash \phi \boxright \psi \tag{9}$$

For every state $a \vdash \phi$ of type t such that $a \trianglelefteq g$ for some state g of type t, there is a state $b \vdash \phi$ of type t such that $b \trianglelefteq a$ and, for every state $c \vdash \phi$ of type t, if $c \trianglelefteq b$, then $c \vdash \psi$. (10)

Proof. Suppose (9). By lemma 4.2 and the fact that x^* is maximal, there is some $d \vdash \phi \wedge \psi$ such that $d \triangleleft e$ for all $e \vdash \phi \wedge \neg\psi$. Now consider any a of the type described. If $d \trianglelefteq a$, then d is the needed b, so suppose otherwise. In that case, $a \triangleleft d$ (because $\trianglelefteq$ is connected), and so $a \triangleleft e$ (because $\triangleleft$ is transitive by observation 2.5). But then a is itself the requisite b (since $\trianglelefteq$ is reflexive), and so (10).

For the other direction, suppose (10) and consider any $a \vdash \phi$. What we are going to show is that there is always a $b \vdash \phi$ such that $x^* \vdash a \vee b \boxright \psi$. But in that case, we can use A5 to disjoin the antecedents of all such counterfactuals, with the result being a disjunctive antecedent that is logically equivalent to ϕ. So $x^* \vdash \phi \boxright \psi$ by substitution, and therefore (9) by observation 4.1.

Suppose then that $a \vdash \phi$ and that there is no d such that $a \trianglelefteq d$. In that case, we have $x^* \vdash a \vee a \boxright \neg a$ and so $x^* \vdash a \vee a \boxright \psi$ by (B8), and so a itself can be the requisite b. Now suppose instead that $a \vdash \phi$ and that there is some d such that $a \trianglelefteq d$. It thus follows that there is some b as descried in (10) and, furthermore, $b \models \psi$ because $\trianglelefteq$ is reflexive. There are then two cases. If $a \trianglelefteq b$, then $a \vdash \psi$, and so $x^* \vdash a \vee b \boxright \psi$ by A1 and E1. On the other hand, if $a \ntrianglelefteq b$, then $x^* \vdash a \vee b \boxright b$ and so $x^* \vdash a \vee b \boxright \psi$. The upshot is that for any $a \models \phi$, there is a $b \vdash \phi$ such that $x^* \vdash a \vee b \boxright \psi$, as claimed. ■

Lemma 4.4 (Truth Lemma). *Let ϕ be a sentence of type t and $\mathcal{M}^t$ the canonical model of that same type. Then for all $x \in W^{m,n}$:*

$$\mathcal{M}^t, x \models \phi \textit{ iff } x \vdash \phi$$

Proof. The proposition holds for atomic sentences by construction and, whenever it holds for a set of sentences, it also holds for the truth functional compounds of those sentences. The proof thus reduces to the case in which ϕ has the form $\gamma \boxright \psi$. That it holds in this case follows from proposition 4.3 and lemma 4.3 by induction and the construction of $\mathcal{M}^t$. ■

Theorem 4.1 (Completeness)**.** *If ϕ is a consistent sentence of **B2**, then ϕ has a connected standard model $\mathcal{M}$.*

Proof. By the preceding. ■

Theorem 4.2 (Decidability)**.** ***B2** is decidable.*

Proof. Take any sentence ϕ of $\mathcal{L}$. The type t of that sentence can be determined recursively, and the corresponding canonical model $\mathcal{M}^t$ is finite, since the set of all states of type t is finite. Given theorem 3.2 and lemma 4.4, we can thus determine whether $\vdash \phi$ by recursively checking to see if ϕ is provable by recursively checking to see if ϕ is true at every world in $\mathcal{M}^t$. ■

References

Berntson, Daniel. 2020. “The Paradox of Counterfactual Tolerance.” June 26.

Burgess, John P. 1981. “Quick Completeness Proofs for Some Logics of Conditionals.” *Notre Dame Journal of Formal Logic* 22, no. 1 (January): 76–84.

Chandler, Hugh S. 1976. “Plantinga and the Contingently Possible.” *Analysis* 36 (2): 106–109.

Fine, Kit. 2012. “Counterfactuals without Possible Worlds.” *The Journal of Philosophy* 109 (3): 221–246.

Henkin, Leon. 1949. “The Completeness of the First-Order Functional Calculus.” *Journal of Symbolic Logic* 14, no. 3 (September): 159–166.

Lewis, David K. 1971. “Completeness and Decidability of Three Logics of Counterfactual Conditionals.” *Theoria* 37 (1): 74–85.

———. 1973. *Counterfactuals.* 278–281. 2. Blackwell.

Pollock, John L. 1975. “Four Kinds of Conditionals.” *American Philosophical Quarterly* 12 (1): 51–59.

———. 1976a. *Subjunctive Reasoning.* Reidel.

———. 1976b. “The ‘Possible Worlds’ Analysis of Counterfactuals.” *Philosophical Studies* 29, no. 6 (June): 469–476.

Salmon, Nathan Ucuzoglu. 1979. “How Not to Derive Essentialism from the Theory of Reference.” *The Journal of Philosophy* 76, no. 12 (December): 703.

Stalnaker, Robert C. 1968. “A Theory of Conditionals.” In *Studies in Logical Theory (American Philosophical Quarterly Monographs 2),* edited by Nicholas Rescher, 98–112. Blackwell.

Williamson, Timothy. 1992. “Inexact Knowledge.” *Mind* 101 (402): 217–242.

———. 2000. *Knowledge and Its Limits.* 460–464. 297. Oxford University Press.